

## Chapter 1

### Frustration and Fluctuations in Systems with Quenched Disorder

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As Phil Anderson noted long ago, frustration can be generally defined by measuring the fluctuations in the coupling energy across a plane boundary between two large blocks of material. Since that time, a number of groups have studied the free energy fluctuations between (putative) distinct spin glass thermodynamic states. While upper bounds on such fluctuations have been obtained, useful lower bounds have been more difficult to derive. I present a history of these efforts, and briefly discuss recent work showing that free energy fluctuations between certain classes of distinct thermodynamic states (if they exist) scale as the square root of the volume. The perspective offered here is that the power and generality of the Anderson conception of frustration suggests a potential approach toward resolving some longstanding and central issues in spin glass physics.

#### 1. Phil Anderson and Spin Glass Theory

It is a great pleasure, both personally and scientifically, to contribute to this volume in honor of Phil Anderson's 90th birthday. The importance and influence of Phil's research in shaping the modern field of condensed matter physics (including coining the term, along with Volker Heine) is widely recognized. There are few currently active, fruitful areas of condensed matter research that have not been either created or (at least) strongly influenced by Phil. His influence, moreover, is not limited to condensed matter physics: he pointed the way<sup>1</sup> toward what is now universally known as the Higgs mechanism; set the stage for later developments in complexity science by emphasizing the importance of a nonreductionist scientific viewpoint<sup>2</sup> (not a widely held view at the time); and later explored and emphasized the connections between the statistical mechanics of quenched disorder and problems in biology, graph theory, and other areas outside of physics.<sup>3</sup>

It is the topic of quenched disorder that I will address here. I will not discuss its applications to other areas, but will instead return to the theory's roots. The

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subject of disordered systems — in particular glasses and spin glasses — has been a longstanding interest of Phil's, and one in which he has made numerous fundamental contributions. At the height of his interest, he identified the problem of understanding the physics of structural glasses and their magnetic counterparts, spin glasses, as one of the central unsolved problems in condensed matter physics.<sup>4</sup> His views may well have changed in the ensuing decades, but one can safely argue that understanding the effect of quenched randomness on the condensed state has presented one of the most persistent and confounding set of problems in modern condensed matter physics.

On the subject of spin glasses proper, Phil's contributions are too numerous to list, but it would be a dereliction of duty not to mention two of his most foundational papers. The first is well-known: his 1975 paper<sup>5</sup> with Sam Edwards that swept away numerous distracting details and identified quenched, conflicting ferromagnetic and antiferromagnetic interactions as the ultimate microscopic basis of spin glass behavior. Edwards and Anderson (hereafter referred to as EA) used this idea to propose a simplified model Hamiltonian that has since formed the basis of most theoretical investigations (we include here studies of the Sherrington-Kirkpatrick model,<sup>6</sup> which is an infinite-range version of the EA model). The other important idea proposed in the EA paper concerned the nature of the spin glass order parameter, but that's less relevant to the discussion below.

The second of these papers, not as widely known or cited, concerns Phil's joint (with Gérard Toulouse) introduction of the concept of frustration. The story goes that Gérard attended a lecture in 1976 in which Phil wrote on the blackboard, "The name of the game is frustration." Whether this was elaborated on in the talk I couldn't say, and the principals will have to provide the details — if they remember. But it would have been characteristic of Phil to make this cryptic remark without elaboration and move on. Inspired, Gérard published a classic paper the next year<sup>7</sup> that remains the canonical definition — both conceptual and operational — of frustration. In this formulation, one considers the spins at lattice sites that form a closed loop on a lattice. If there are an odd number of antiferromagnetic couplings on the edges constituting the loop, the spins cannot be arranged to satisfy all of the interactions.

The following year, Phil published an alternative, and more general, definition of frustration<sup>8</sup> in which one studies the free energy fluctuations of two blocks of material (glass, spin glass, ferromagnet, what have you) that have independently relaxed to their respective ground states. I will elaborate both on this and the Toulouse definition of frustration in Sect. 3. For now I will just note that this latter approach has not received as much attention as Toulouse's, but nonetheless — as usual for Phil — it is enormously prescient. In fact, I will argue below that this alternative approach to frustration may, after a long period of dormancy, contain just the right perspective to resolve some longstanding open questions at the heart of spin glass physics.

But for now, the main point is that in both cases frustration arises when a system contains many fixed, conflicting internal constraints, not all of which can be simultaneously satisfied. Like art, physics can sometimes imitate life.

The final topic of this informal account concerns Phil's role in the naming of spin glasses. Phil didn't name them himself, although he clearly was involved.<sup>9</sup> Regardless of the details, he was as usual present at the creation, and provides an amusing discursion on the early days. Most accounts, including Phil's, credit Bryan Coles with inventing the term "spin glasses", although details among the accounts vary somewhat. A competing (and in my opinion, more ungainly) term that had gained some traction at the time was "mictomagnetism", and we might well today be referring to mictomagnets rather than spin glasses.<sup>a</sup> Phil's dry sense of humor led to a disquisition on the etymology of the term, and one cannot do justice to his description other than quoting it in full:<sup>9</sup>

"A few weeks ago I received a letter from Ralph Hudson of the NBS objecting to this term, on the basis that he thought that the only other word in the English language using the same root was 'micturation' and the root was Latin for 'urine'. I think myself that the term is very descriptive: back in the Middle West we used to refer to something as 'p—poor' if it was not worth anything more substantial, and that is a good description of this kind of magnetism. The remanence is small and sluggish, there are peculiar training phenomena, and the susceptibility is often very history dependent. Unfortunately, I am assured by Collin Hurd and by the OED that Hudson is incorrect and that 'micto' is a legitimate Greek root meaning 'mixed'. "

Those interested in Phil's first-hand perspective of the early days of spin glass research should consult his series of *Physics Today* "reference frame" articles<sup>10–16</sup> that helped bring the subject to the attention of the broader physics community.

## 2. Free Energy Fluctuations in the Random Field Ising Model

Systems with quenched disorder possess several features distinguishing them from homogeneous systems. Our focus is on one of these: their energies and free energies are random variables depending on the disorder, and their concomitant fluctuations contain information that can potentially resolve central open questions that remain intractable to this day. These questions include the conjectured multiplicity of pure and ground thermodynamic states, the relations between such distinct states (if they exist), the geometry and energy scaling of their relative interfaces, and so on.

To illustrate the potential usefulness of the information provided by these fluctuations, we turn briefly to a different system: the random-field Ising model (RFIM), which is a uniform Ising ferromagnet subject to a random external field. It can be

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<sup>a</sup>In fact, the term still survives, and you can google it, although Google will insist that you must have meant "micromagnets." Don't give in.

modelled using the Hamiltonian

$$\mathcal{H}_h = - \sum_{\langle x,y \rangle} \sigma_x \sigma_y - \epsilon \sum_x h_x \sigma_x. \quad (1)$$

Here  $x$  and  $y$  are lattice sites in the  $d$ -dimensional cubic lattice  $\mathbf{Z}^d$ ,  $\sigma_x = \pm 1$  is an Ising spin at site  $x$ , the first sum is over nearest-neighbor pairs of sites only, and the fields  $h_x$  are independent, identically distributed random variables representing local external fields acting independently at each site  $x$ . For simplicity, we take the probability distribution of the  $h_x$ 's to be Gaussian with mean zero and variance one. The subscript  $h$  on the LHS of (1) refers to a particular realization of the  $h_x$ 's.

One can now ask whether the ferromagnetic ground state is unstable to breakup by the random field. The answer is clearly yes if  $\epsilon$  is sufficiently large. But is it true for *any* nonzero  $\epsilon$ ?

For a uniform field this question is trivial: in any dimension, a field of any fixed, nonzero magnitude determines the magnetization direction at all temperatures, and so there is no phase transition. A simple scaling argument explains why. In zero field below  $T_c$ , consider the positively magnetized (i.e., “up”) phase. Now apply a small uniform field of magnitude  $h$  pointing down. Overturning a compact patch (or “droplet”) of spins of length scale  $L$  to align with the field is energetically favorable for sufficiently large  $L$ : for Ising spins, the cost in surface energy is of order  $L^{d-1}$  while the lowering of bulk energy is of order  $hL^d$ . So, no matter how small  $h$  is, the system can lower its energy by overturning a sufficiently large droplet. At positive temperature, overturning droplets of spins is certainly entropically favorable as well. Consequently, in any dimension and at any temperature (including zero), there is a unique Gibbs state in an external uniform field of any nonzero magnitude.

Now consider the case when quenched disorder is present, i.e., when the field is random. This requires a modification of the above argument, which was provided in 1975 by Imry and Ma.<sup>17</sup> The boundary energy, which depends only on the ferromagnetic couplings, is unchanged. The bulk energy, however, is determined by the fields, which now fluctuate from region to region. Nevertheless, in an arbitrary large droplet containing  $L^d$  spins, the central limit theorem requires that the typical bulk energy scales as  $L^{d/2}$ . So in two dimensions the competing boundary and bulk energies scale similarly with volume. Imry and Ma concluded that in two dimensions and below, the ferromagnetic ground state should be unstable for any nonzero  $\epsilon$ , while above two dimensions (and with small but nonzero  $\epsilon$ ) ferromagnetic long-range order persists. (In fact, Imry and Ma mainly focused on continuous spin models, where the boundary energy scales as  $L^{d-2}$ , giving a lower critical dimension of 4.)

Normally, such an argument would be sufficient to settle the matter, but a few years later a more detailed field-theoretical analysis based on supersymmetry<sup>18</sup> concluded that the critical behavior of a  $d$ -dimensional spin system in a random external field is equivalent to that of the corresponding  $(d-2)$ -dimensional system in the *absence* of an external field. This “dimensional reduction” argument therefore predicts the lower critical dimension of the RFIM to be three.

This controversy was eventually resolved by rigorous mathematical arguments, first by Imbrie<sup>19</sup> and later by Aizenman and Wehr.<sup>20,21</sup> Imbrie proved that the Ising model in a random magnetic field in three dimensions exhibits long-range order at zero temperature and sufficiently small disorder, indicating that the lower critical dimension of the RFIM is indeed two. Aizenman and Wehr later proved that in two dimensions at all temperatures and fields, the RFIM possesses a unique Gibbs state.

This now ancient controversy is recounted for two reasons. First, it represents an interesting — and rare — example where rigorous mathematics resolved an open and important controversy in theoretical (and indeed, experimental) physics. It may be the case that something similar will be required for spin glasses and possibly even structural glasses, where the most basic questions have persisted as a subject of intense controversy over decades.

More relevant to this paper, though, is that Aizenman and Wehr essentially made the Imry-Ma argument rigorous by analyzing fluctuations (with respect to the quenched disorder) of the free energy difference between putative positively and negatively magnetized states. Although the RFIM is not a frustrated system, the spirit of the Aizenman-Wehr method aligns with the Anderson approach to characterizing and understanding frustration.

### 3. Frustration

We turn now to finite-dimensional spin glasses, where almost all of the basic questions remain open. These include whether an equilibrium phase transition occurs above some dimension; if so, the nature of the broken symmetry (if any) of the spin glass phase; whether (up to global symmetry transformations) the spin glass phase is unique; if not, the nature of the relationships among the many spin glass phases; whether there exists an upper critical dimension above which mean field theory holds;<sup>b</sup> and numerous others. And this list doesn't include questions concerning the nonequilibrium dynamical behavior of spin glasses, which won't be addressed in this paper.

For concreteness we confine our attention to nearest-neighbor models defined by the EA Hamiltonian:<sup>5</sup>

$$\mathcal{H}_{\mathcal{J}} = - \sum_{\langle x,y \rangle} J_{xy} \sigma_x \sigma_y - h \sum_x \sigma_x , \quad (2)$$

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<sup>b</sup>In homogeneous systems, this question usually refers only to behavior at or near the critical point. For spin glasses it is considerably more far-reaching. Here we're asking whether the *low-temperature* properties — i.e., the order parameter and the nature of broken symmetry — corresponds in *any* finite dimension to the replica symmetry breaking<sup>22–26</sup> that occurs in the low-temperature phase of the infinite-range Sherrington-Kirkpatrick model.<sup>6</sup> This is generally not an issue in homogeneous systems, where mean-field theory usually provides a useful guide to the nature of the low-temperature phase well below  $T_c$  in any dimension where a phase transition occurs.

where  $x$  and  $y$  are sites in the  $d$ -dimensional cubic lattice,  $\sigma_x = \pm 1$  is the Ising spin at site  $x$ , the couplings  $J_{xy}$  are independent, identically distributed random variables,  $\mathcal{J}$  denotes a particular realization of the couplings (corresponding physically to a specific spin glass sample with quenched disorder),  $h$  is an external magnetic field, and the first sum is over nearest neighbor sites only. We hereafter take  $h = 0$  and the spin couplings  $J_{xy}$  to be symmetrically distributed about zero; consequently, the EA Hamiltonian in (2) possesses global spin inversion symmetry.

A striking feature of the EA Hamiltonian is the presence of *frustration*, meaning the inability of any spin configuration to simultaneously satisfy all couplings. It is easily verified that, in any dimension larger than one, all of the spins along any closed circuit  $\mathcal{C}$  in the edge lattice cannot be simultaneously satisfied if

$$\prod_{\langle x,y \rangle \in \mathcal{C}} J_{xy} < 0. \quad (3)$$

This definition of frustration is due to Toulouse.<sup>7</sup>

Toulouse's geometry-based definition is appealing on several levels, and has been the starting point for numerous investigations (see, for example, Refs. [27,28]). It provides a simple test to determine whether a given type of spin system possesses frustration, and suggests the underlying reason why certain systems may possess multiple pure or ground states. More generally, the quantification of conflicting internal constraints provides a powerful conceptual tool for understanding certain general aspects of complex behavior in the broader study of complex systems.

Its drawback is that in some sense the Toulouse approach is *too* well-suited to spin glasses; it is difficult to see how it can be generalized in a natural way to non-spin systems that surely possess frustration, such as structural glasses or combinatorial optimization problems. For these systems the Anderson definition of frustration is more useful; it is sufficiently general that (with minor modification as needed) it should apply to any system. As we will see, it also provides a conceptual starting point for mathematical studies that hold promise for resolving the long-controversial issue of pure/ground state multiplicity in frustrated systems with quenched disorder.

The idea itself is rather simple, although its simplicity conceals a profound and very useful insight. Based on a preliminary study of Anderson and Pond,<sup>29</sup> Anderson proposed<sup>8</sup> considering the free energy fluctuations of two statistically identical blocks of the same material. It is simplest to describe the procedure at zero temperature, although it is easily modified for positive temperature. So let each block of material independently relax to its ground state<sup>c</sup>. One then brings the two blocks together and measures the fluctuations in the coupling energy across their interface. In nonfrustrated systems, such as ferromagnets, the energy fluctuations scale as the surface area  $A$  of contact. This scaling holds for both homogeneous

<sup>c</sup>For a finite system with specified boundary conditions, such as periodic, and continuous disorder, such as Gaussian, the ground state is unique up to a global symmetry.

and random ferromagnets (in which the bond strengths are positive, i.i.d. random variables).

However, if frustration is present, then it will be the case that

$$\lim_{A \rightarrow \infty} \langle E^2 \rangle / A^2 = 0. \quad (4)$$

In fact, one can turn this around and use (4) as the general definition of frustration, which we will hereafter do.

Using reasoning based on the central limit theorem Anderson further conjectured that

$$\lim_{A \rightarrow \infty} \langle E^2 \rangle / A = O(1). \quad (5)$$

That is, in a frustrated system one might expect the energy fluctuations of the ground states to scale as the *square root* of the surface area of contact. However, (5) is a rough estimate.

The definition (4) of frustration, although not as widely known or appreciated as (3), contains the seeds of a powerful approach to understanding realistic spin glasses and other complex systems. We turn now to a natural outgrowth of this approach, namely, scaling theories of the spin glass phase.

#### 4. Scaling Theories of the Spin Glass Phase

The idea of investigating fluctuations of free energy differences in the presence of frustration leads naturally to a scaling approach for understanding the low-temperature spin glass phase. This approach, which has a long history in the study of phase transitions and broken symmetry, examines how the “stiffness” of the low-temperature phase scales with the system size  $L$ . A stable phase requires the stiffness — roughly speaking, the free energy cost associated with overturning a droplet of spins — to increase (or at least not decrease) with  $L$ , usually as a power law, although other forms are possible in principle.

This approach as applied to spin glasses began with the early work of Anderson and Pond,<sup>29</sup> and was developed throughout the 1980's.<sup>30–38</sup> The essential idea is to study the fluctuations in a finite volume of the spin glass free energy as one changes boundary conditions, for example from periodic to antiperiodic. Physically, such a change in boundary conditions generates relative interfaces inside the box, so one is effectively studying the interface free energy. Given the close relation between this approach and that of the Anderson definition of frustration, one might expect that the presence of frustration will generate profound effects on such interfacial free energies. And of course it does.

##### 4.1. Interface Geometry

Before proceeding, some remarks are necessary concerning the relation between this procedure and the presence of many states. While switching from periodic to an-

tipperiodic boundary conditions always generates relative interfaces, geometrically one of three things can happen. Consider a “window”<sup>39,40</sup> of large but fixed linear size  $w$  centered at the origin, and consider the interfaces generated when  $L \gg w$ . The first possibility is that as  $L$  grows increasingly larger (with  $w$  fixed), the interfaces eventually move outside of the window, so that the thermodynamic state *inside* the window is the same for both periodic and antiperiodic boundary conditions. If this is the case, then there are only two spin glass pure states (or ground states at zero temperature), which are global flips of each other.

The other possibility, of course, is that no matter how far away the boundaries move, interfaces always penetrate inside the window. This is the signature of multiple spin glass pure state pairs. This possibility can be further divided into two parts: either the interfaces have vanishing density as the window size increases (with the order of limits being  $L \rightarrow \infty$  followed by  $w \rightarrow \infty$ ) or else the interface density remains bounded away from zero. The former zero-density case is what one finds in the ferromagnet, which exhibits  $d - 1$ -dimensional interfaces in a  $d$ -dimensional system. In contrast, the latter “space-filling” case, if it occurs, requires  $d$ -dimensional interfaces within a  $d$ -dimensional system, which would signify a novel feature of spin glasses. Huse and Fisher<sup>41,42</sup> refer to the zero-density situation as “regional congruence”: the states are locally the same almost everywhere. The more interesting situation with space-filling interfaces was denoted “incongruence”: the states, although similar in a statistical sense, are dissimilar everywhere. It was proven in<sup>43</sup> that any procedure using boundary conditions chosen in a coupling-independent manner (as in the periodic-antiperiodic situation above) always results either in a single pair of states (no interfaces in the window)<sup>d</sup> or else many incongruent pairs of states. Regionally congruent states, should they exist, can only be generated using coupling-*dependent* boundary conditions, requiring procedures as yet unknown. In what follows we therefore confine the discussion to incongruent states.

#### 4.2. Interface Energetics

The preceding discussion focuses exclusively on the geometry of interfaces between pure states; we now discuss their energetics. In a ferromagnet, whether homogeneous or random, all couplings have the same sign, so the interface energy scales with the number of edges it comprises. In a spin glass, interface energetics remain an open problem, and a very important one: if one knows how the interface energy scales with its size, one can finally resolve the longstanding open question regarding the multiplicity of pure states in the spin glass phase.

Here’s why. By the same reasoning that led to the Anderson definition of frus-

<sup>d</sup>An absence of interfaces within the window implies that all interfaces necessarily generated by changing boundary conditions must be zero-density, since positive density interfaces *must* penetrate the window.<sup>44</sup> The difference between this case and the regionally congruent case is that in the latter, zero-density interfaces continue to penetrate the window no matter how far away the boundaries are, while in the former, the interfaces deflect to infinity as the boundaries move out to infinity.



tration, it is reasonable to expect that the free energy of a space-filling interface of linear extent  $L$  should scale no faster than  $L^{d/2}$ . However, it is possible that correlations could lower the minimal interface free energy, so that it scales as  $L^\theta$ , with  $0 \leq \theta \leq d/2$  (assuming a stable spin glass phase). The lower bound of  $\theta = 0$  is predicted<sup>e</sup> by the mean-field replica symmetry breaking (RSB) picture of the spin glass phase,<sup>22–26</sup> while  $0 < \theta < (d-1)/2$  is predicted by the chaotic pairs picture.<sup>40,45,46</sup> Both are many-states pictures, though with very different thermodynamics and organization of the incongruent pure states, as will be discussed below.

Based on numerical results and scaling arguments, Fisher and Huse<sup>35</sup> conjectured a stronger upper bound than  $L^{d/2}$ ; they argued that in fact  $\theta \leq (d-1)/2$ . If this is correct, then the question of whether incongruent states exist reduces to the question of whether their interface free energy should scale faster or slower than  $L^{(d-1)/2}$ . Fisher and Huse argued, along roughly similar lines to Anderson, that it should scale as  $L^{d/2}$ . The resulting contradiction between the upper and lower bounds leads to the conclusion that incongruent states cannot exist in the EA spin glass in any finite dimension.

But is the conjectured upper bound  $\theta = (d-1)/2$  correct? This was initially a matter of some controversy, but within a few years it was proved, using rigorous mathematical arguments, by Aizenman and Fisher,<sup>47</sup> and independently and a little later by Newman and Stein.<sup>48</sup> Unfortunately, neither was ever published, but the arguments are now familiar to those who work in this area. I will informally sketch the basic idea of the proof here. First, let me state the exact result more formally:

**Theorem 1.** *Let  $F_P$  be the free energy of the finite-volume Gibbs state generated by Hamiltonian (2) (with  $h = 0$ ) in a box  $\Lambda$  of volume  $L^d$  using periodic boundary conditions, and  $F_{AP}$  that generated using antiperiodic boundary conditions. Let  $X_\Lambda = F_P - F_{AP}$ . Then  $\text{Var}(X_\Lambda) \leq \text{const.} \times L^{d-1}$ , where  $\text{Var}(\cdot)$  denotes the variance over all of the couplings inside the box.*

Physically, this implies that the fluctuations in free energy (and therefore the interface free energy) induced by changing boundary conditions from periodic to antiperiodic in a box of volume  $L^d$  scales as  $L^{(d-1)/2}$ . The theorem as stated above is more restrictive than necessary; the same result applies for any two boundary conditions that are *gauge-related*, i.e., that can be transformed into each other by reversing the sign of some subset of couplings on the boundary of the box.<sup>f</sup> So, for example, the same result holds for any two distinct fixed boundary conditions.

The proof uses a *martingale decomposition* of the free energy difference, as follows. First note that, by the gauge-relatedness of the two boundary conditions,

<sup>e</sup>More precisely, this scaling applies to interfaces between pairs of incongruent states within the *same* thermodynamic state. Interfaces between incongruent states belonging to *different* thermodynamic states, which would be expected within the RSB picture upon switching from periodic to antiperiodic boundary conditions, would presumably have  $\theta > 0$ . For a discussion of thermodynamic states within RSB, see [46], Sect. 7.9.

<sup>f</sup>A coupling on the boundary of the box connects a site inside  $\Lambda$  to one on the boundary of  $\Lambda$ .

$E[X_\Lambda] = 0$ , where  $E[\cdot]$  denotes a full average over all the couplings inside the box. Next number each coupling inside the box, as shown in Fig. 1. We now successively

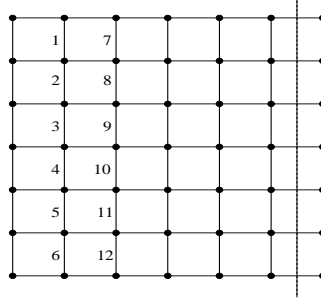


Fig. 1. Schematic of a box with each coupling numbered (only the first twelve are explicitly indicated). The dashed line on the right indicates couplings whose transformation  $J_{xy} \rightarrow -J_{xy}$  is equivalent to switching from periodic to antiperiodic boundary conditions, and which according to the argument in the text are the only ones contributing to the variance.

average the free energy difference  $X_\Lambda$  over an increasing number of couplings. Let  $x_{\Lambda,j} = E[X_\Lambda | b_1 b_2 \dots b_j]$ , where  $E[A | y_1, y_2, \dots, y_k]$  is the conditional expectation, or average, of  $A$  conditioned on the random variables  $y_1, y_2, \dots, y_k$ . That is, suppose that  $A$  is a random quantity depending on the  $N$  random variables  $y_1, y_2, \dots, y_N$ . Then  $E[A | y_1, y_2, \dots, y_k]$  represents the quantity resulting from averaging  $A$  over  $y_{j+1}, y_{j+2}, \dots, y_N$ . So  $x_{\Lambda,0}$  represents  $X_\Lambda$  fully averaged over all of the couplings in the interior of the box, and  $x_{\Lambda,N} = X_\Lambda$ , the original unaveraged free energy difference.

If there are  $N$  couplings inside the box, it is not hard to see that

$$\text{Var}(X_\Lambda) = \text{Var}\left[\sum_{j=0}^{N-1} (x_{\Lambda,j+1} - x_{\Lambda,j})\right] = \sum_{j=0}^{N-1} \text{Var}(x_{\Lambda,j+1} - x_{\Lambda,j}), \quad (6)$$

where the second equality follows because the so-called martingale differences  $x_{\Lambda,j+1} - x_{\Lambda,j}$  are orthogonal quantities.<sup>§</sup>

Now suppose that the antiperiodic boundary conditions are applied to the right and left boundaries, and consider an  $x_{\Lambda,j}$  conditioned on any subset of the couplings *except* those cut by the dashed line in Fig. 1. Any such  $x_{\Lambda,j} = 0$ , because one is averaging over all of the couplings cut by the dashed line, and taking  $J_{xy} \rightarrow -J_{xy}$  over these couplings is equivalent to switching between periodic and antiperiodic boundary conditions. The only nonzero contributions to the sum in (6) therefore comes from conditioning on these boundary couplings, each of which contributes a term of order one to the variance. This completes the argument.

<sup>§</sup>Two random variables  $A$  and  $B$  are orthogonal if  $E[AB] = 0$ . It is easy to see that this holds for any two martingale differences.

## 5. Interface Free Energy Fluctuations

If we can now find a strong *lower* bound for interface free energies between incongruent states, we would be in a position to determine whether such states can exist at all; and if they do, what their properties should be. Unfortunately, finding a lower bound is considerably more difficult than finding an upper bound, for reasons that will be discussed below. It has been almost 25 years since the upper bound was proved, and no progress on finding a lower bound has been made until very recently.

Unlike the upper bound, which is scenario-independent, construction of a lower bound requires a specific picture of the spin glass phase. In other words, one first needs to ask: what's doing the fluctuating? We will examine here four distinct pictures that have been proposed, each of which gives a different answer to the question.

### 5.1. Scenarios for the Low-Temperature Spin Glass Phase

Probably the most familiar are the *mixed-state* pictures, in which the spin glass phase consists of infinitely many thermodynamic states, each of which is itself a mixture of infinitely many incongruent pure state pairs with nonzero weights (within their respective thermodynamic states). The mean-field-inspired RSB scenario is such a picture, although others are also possible. Recall from Sect. 4.2 that in these scenarios the smallest interface free energies remain order one independently of the interface size ( $\theta = 0$ ).

Almost equally familiar are the scaling/droplet pictures discussed in Sect. 4. These are two-state pictures with  $\theta > 0$  and in which no interface appears in a window far from the boundaries when one switches from periodic to antiperiodic boundary conditions.

There are two other less familiar pictures that should nevertheless be included: the “TNT” picture of Krzakala-Martin<sup>49</sup> and Palassini-Young,<sup>50</sup> and the chaotic pairs picture discussed in Sect. 4.2. If one constructs a  $2 \times 2$  grid listing the different possibilities for interface geometry (space-filling vs. zero-density) and energetics ( $\theta = 0$  vs.  $\theta > 0$ ) then these additional pictures are required for completeness.<sup>51h</sup> The TNT picture is presumably a two-state picture<sup>43</sup> with zero-density interfaces and  $\theta = 0$ , while chaotic pairs is a many-state picture with space-filling interfaces and  $\theta > 0$ . Unlike RSB, it is not a (nontrivial) mixed-state picture: while it contains infinitely many distinct thermodynamic states, each one consists of a single spin-reversed pure state pair.

So of these four, only RSB and chaotic pairs possess incongruent states, and any lower bound on interface free energies can help to determine whether they are likely candidates for the spin glass phase, or else are not allowed at all. How so? Consider again the energetics of interfaces between putative incongruent pure

<sup>h</sup>They also arise naturally from a metastate analysis of possible spin glass phases.<sup>40,45,52,53</sup>

states. As already mentioned, unlike in the ferromagnet, whether homogeneous, random-bond, or random-field, the sign of the free energy difference between two putative spin glass states varies as one moves along their relative interface. If the single-coupling energy differences are independent, then one expects an energy that varies as the square root of the number of couplings in the interface — i.e., in a volume of size  $L$ , the fluctuations in the interface free energy would scale as  $L^{d/2}$ , as in the conjecture accompanying the Anderson definition of frustration. If this were indeed the case, such fluctuations would violate, in any finite dimension, the upper bound of  $L^{(d-1)/2}$  described in Sect. 4.2, and the existence of incongruent states would therefore be impossible in finite-dimensional spin glasses.

But the single-coupling free energy differences are certainly *not* independent. Whether the correlations in free energy fluctuations as one moves along the interface are strong enough to decrease the free energy scaling exponent from its independent value is the crucial question that determines whether incongruent states are present in spin glasses — or not. Hence, determining an accurate lower bound is of central importance for resolving the question of multiplicity of states in realistic spin glasses.

## 5.2. Lower Bound

If it's so important, why has it taken so long to find a lower bound for free energy difference fluctuations? Progress has been held back by several technical hurdles; the two most troublesome are the “cancellation problem” and the “identification problem”.

The cancellation problem has already been discussed in Sect. 4.2. If incongruent states exist, then an arbitrarily chosen coupling will have, with positive probability, a free energy difference of order one between the two states. But if one uses the usual techniques, such as martingale differences, to extrapolate to the entire volume, cancellations between the many terms lead to an ambiguous outcome.

Equally difficult is the identification problem. It is hard to see how one can estimate the extent of free energy difference fluctuations between two thermodynamic states  $\Gamma$  and  $\Gamma'$  without averaging over the couplings inside the volume. But as one does so, what happens to the original states? Unlike in ferromagnets and other homogeneous systems, there is no clearcut connection between boundary conditions and thermodynamic states, and (if there are many incongruent states) the states themselves can change as one varies the couplings inside the box. So *a priori* it is not even clear what one is calculating during the averaging procedure.

In a very recent paper,<sup>54</sup> these and other problems were finally surmounted, although (at the moment) for a limited class of incongruent states. For these incongruent states it was found that the fluctuations in free energy differences do indeed scale as  $L^{d/2}$ ; or more formally, the variance of the free energy difference between the incongruent states considered scales linearly with the volume.

If these results can be extended to the set of incongruent states in general, does

one then have a contradiction? In a strict mathematical sense, not quite yet.<sup>i</sup> The problem is that the quantity for which the lower bound is derived is not exactly the same as that for which the upper bound was derived, although both are just different representations of the free energy difference in a finite volume, and so are equivalent in a physical sense.

The upshot is that we may be on the cusp of resolving the problem of multiplicity of pure states in realistic spin glasses — but at the moment it is unclear as to whether the results can be extended to bring the upper and lower bounds into alignment. Whether this is eventually done or not, it is clear that the insights that Phil Anderson had thirty-five years ago into the nature of frustration are still actively guiding and influencing fundamental research today.

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<sup>i</sup>Whether such a result already provides a sufficient heuristic or theoretical physics-style argument for the nonexistence of incongruent states is left up to the reader.

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